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Effect of time-periodic boundary temperatures on the onset of double diffusive convection in a horizontal anisotropic porous layer

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Abstract

The effect of time-periodic boundary temperatures on the onset of double diffusive convection in a fluid-saturated anisotropic porous medium is studied by making a linear stability analysis. A perturbation method based on small amplitude of the imposed temperature modulation is used to compute the critical values of thermal Rayleigh number and wave number. The correction thermal Rayleigh number is calculated as a function of frequency of modulation, viscosity ratio, anisotropy parameter, porous parameter, Prandtl number, diffusivity ratio and solute Rayleigh number. The effect of various physical parameters is found to be significant at moderate values of the frequency. We found that it is possible to advance or delay the onset of double diffusive convection by proper tuning of the frequency of modulation of the wall temperature. The effect of various parameters on the stability of the system is brought out.

Keywords: Double diffusive convection; Modulation; Anisotropy; Porous medium

1. Introduction

The problem of double diffusive convection in porous media has attracted considerable interest in recent time because of its wide range of applications, from the solidification of binary mixtures to the migration of solutes in water-saturated soils. Other examples include geophysical systems, electro chemistry, the migration of moisture through air contained in fibrous insulation.

Early studies on the phenomena of double diffusive convection in porous media are mainly concerned with problem of convective instability in a horizontal layer heated and salted from below. The onset of double diffusive convection in a horizontal porous layer has been investigated by Rudraiah et al. [1] using non-linear perturbation theory. The linear stability analysis of the thermohaline convection is carried out by Poulikakos [2] using the Darcy–Brinkman model. The double diffusive convection in a porous media in the presence of cross-diffusion effects is analysed by Rudraiah and Malashetty [3].

One of the effective mechanism to control convection is by maintaining a non-uniform temperature gradient across the boundaries. The non-uniform temperature gradient may be generated by (i) appropriate heating or cooling at the boundaries (ii) through flow (iii) appropriate distribution of heat sources and (iv) radiative heat transfer (see e.g. [4]). These are concerned only with space dependent temperature gradient. However, in many practical problems the non-uniform temperature gradient is a function of both space and time. This is to be determined by solving energy equation with suitable time dependent temperature boundary conditions.

There are many studies available in the literature concerning how a time-periodic boundary temperature affects the onset of Rayleigh–Benard convection. Most of the findings related to these problems have reviewed

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Nomenclature

d	height of the porous layer
F	porous parameter, k_z/d^2
$\mathbf{g}$	gravitational acceleration
$\mathbf{k}$	unit vector in the vertical direction
K_m	effective thermal conductivity of porous media
k_x, k_z	permeability components in horizontal and vertical directions
l, m	wave numbers in x-, y-directions
p	pressure
Pr	Prandtl number, ν/κ_T
$\mathbf{q}$	mean filter velocity, (u, v, w)
$\mathbf{Q}$	modified velocity vector, which includes permeabilities, $(u/k_x, v/k_x, w/k_z)$
R	thermal Rayleigh number, $\beta_1 g \Delta T d k_z/\nu\kappa_T$
Rs	solute Rayleigh number, $\beta_2 g \Delta S d k_z/\nu\kappa_T$
S	solute concentration
ΔS	salinity difference between the walls
T	temperature
ΔT	temperature difference between the walls
t	time
x, y, z	space co-ordinates

Greek symbols

α	horizontal wave number
α_c	critical wave number
β_1	thermal expansion co-efficient
β_2	solute expansion co-efficient
γ	heat capacity ratio, $(\rho c_p)_m/(\rho c_p)_f$
δ	porosity
ε	amplitude of modulation
κ_S	solute diffusivity
κ_T	effective thermal diffusivity, $K_m/(\rho c)_f$
μ_e	effective viscosity
μ_f	viscosity of the fluid
ν	kinematic viscosity, μ_f/ρ_R
τ	diffusivity ratio, κ_S/κ_T
ρ	density
ϕ	phase angle
Ω	frequency of the modulation
ω	non-dimensional frequency, $\Omega d^2/\kappa_T$
ξ	anisotropy parameter, k_x/k_z

Other symbols

∇_1^2	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
∇^2	$\nabla_1^2 + \frac{\partial^2}{\partial z^2}$

Subscripts

b	basic state
c	critical
f	fluid
R	reference value

Superscripts

$*$	dimensionless quantity
$\prime$	perturbed quantity

by Davis [5]. In case of small amplitude temperature modulations, a linear stability analysis was performed by Venezian [6]. Rosenblat and Herbert [7], Rosenblat and Tanaka [8] and Roppo et al. [9] have studied the effect of thermal modulation on the onset of convection in a horizontal fluid layer. On the other hand the studies related to the effect of temperature modulation on the onset of convection in a fluid saturated porous medium have received marginal attention. The effect of time-periodic wall temperature on the onset of convection in a porous medium has been studied by Caltagirone [10], Rudraiah and Malashetty [4] and Malashetty and Wadi [11]. Recently, Malashetty and Basavaraja [12,13] have studied the effect of time-periodic temperature modulation on the onset of convection in a horizontal anisotropic porous layer. All these investigations are restricted to a single component fluid and porous layers. To our knowledge the studies on the effect of temperature modulation on the double diffusive convection in a horizontal anisotropic porous layer are not available in the literature.

The main object of this work is to study the effect of time-periodic boundary temperatures on the onset of double diffusive convection in a horizontal anisotropic porous layer. The amplitude and frequency of the modulation are externally controlled parameters and hence the onset of convection can be delayed or advanced by the proper tuning of these parameters. Therefore temperature modulation can be used as a mechanism to delay convection to achieve higher efficiencies in case of material processing applications and advance it for achieving major enhancement of mass, momentum and heat transfer.

2. Mathematical formulation

We consider a fluid-saturated anisotropic porous medium confined between two infinite horizontal surfaces, a distance 'd' apart and a stabilizing uniform concentration gradient and a vertical downward gravity force acts on the fluid. A Cartesian co-ordinate is taken with the origin in the lower boundary and the z-axis vertically upwards. The surface temperatures are time-periodic, externally imposed and are taken as

$$T_R + \frac{\Delta T}{2}[1 + \varepsilon\cos\Omega t] \quad \text{at } z = 0 \tag{1}$$

and

$$T_R - \frac{\Delta T}{2}[1 - \varepsilon\cos(\Omega t + \phi)] \quad \text{at } z = d. \tag{2}$$

A constant salinity gradient ΔS is maintained between the two surfaces. Accordingly

$$S_R + \frac{\Delta S}{2} \quad \text{at } z = 0 \tag{3}$$

and

$$S_R - \frac{\Delta S}{2} \quad \text{at } z = d. \tag{4}$$

The porous medium is assumed to posses horizontal isotropy. With the assumptions and approximations frequently made for the study of double diffusive convection in a porous medium, the basic equations are [14]

$$\nabla \cdot \mathbf{q} = 0, \tag{5}$$

$$\frac{1}{\delta}\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{\delta^2}(\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho_R}\nabla p + \frac{\rho_f}{\rho_R}\mathbf{g} - \frac{\mu_f}{\rho_R}\mathbf{Q} + \frac{\mu_e}{\rho_R}\nabla^2\mathbf{q}, \tag{6}$$

$$(\rho c_p)_m\frac{\partial T}{\partial t} + (\rho c_p)_f(\mathbf{q}\cdot\nabla)T = K_m\nabla^2 T, \tag{7}$$

$$\frac{\partial S}{\partial t} + \frac{1}{\delta}(\mathbf{q}\cdot\nabla)S = \kappa_S\nabla^2 S, \tag{8}$$

$$\rho_f = \rho_R[1 - \beta_1(T - T_R) + \beta_2(S - S_R)]. \tag{9}$$

2.1. Basic state

Basic state of the fluid is quiescent and in the basic state, the temperature T_b, solute concentration S_b, pressure p_b, density ρ_b satisfy the following equations

$$\gamma\frac{\partial T_b}{\partial t} = \kappa_T\frac{\partial^2 T_b}{\partial z^2}, \tag{10}$$

$$\frac{d^2 S_b}{dz^2} = 0, \tag{11}$$

$$-\frac{\partial p_b}{\partial z} = \rho_b g \tag{12}$$

and

$$\rho_b = \rho_R[1 - \beta_1(T_b - T_R) + \beta_2(S_b - S_R)], \tag{13}$$

where $\gamma = (\rho c_p)_m/(\rho c_p)_f$ and $\kappa_T = K_m/(\rho c_p)_f$ (effective thermal diffusivity).

The solutions of Eqs. (10) and (11) subject to the boundary conditions (1)–(4) are

$$T_b = T_R + \frac{\Delta T}{2}\left\{\left(1 - \frac{2z}{d}\right) + \varepsilon\mathrm{Re}\left\{[a(\lambda)e^{\lambda z/d} + a(-\lambda)e^{-\lambda z/d}]e^{-i\Omega t}\right\}\right\}, \tag{14}$$

$$S_b = S_R + \frac{\Delta S}{2}\left(1 - \frac{2z}{d}\right), \tag{15}$$

where

$$\lambda = (1 - i)\left[\frac{\gamma\Omega d^2}{2\kappa_T}\right]^{1/2}, \quad a(\lambda) = \left[\frac{e^{-i\phi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}\right]$$

and Re stands for the real part.

2.2. Linear stability analysis

Let the basic state be disturbed by an infinitesimal thermal perturbation. We now have

$$\mathbf{q} = \mathbf{q}', \quad p = p_b + p', \quad T = T_b + T',$$
$$S = S_b + S', \quad \rho_f = \rho_b + \rho'. \tag{16}$$

The prime indicates that the quantities are infinitesimal perturbations.

Substituting (16) into Eqs. (5)–(9) and using the basic state solutions, we get the linearized equations governing the infinitesimal perturbations in the form

$$\frac{1}{\delta}\frac{\partial \mathbf{q}'}{\partial t} = -\frac{1}{\rho_R}\nabla p' + (\beta_1 T' - \beta_2 S')g\mathbf{k} - v\mathbf{Q}' + \frac{\mu_e}{\rho_R}\nabla^2\mathbf{q}', \tag{17}$$

$$\gamma\frac{\partial T'}{\partial t} + w'\left(\frac{\partial T_b}{\partial z'}\right) = \kappa_T\nabla^2 T', \tag{18}$$

$$\frac{\partial S'}{\partial t} + \frac{1}{\delta}w'\left(\frac{dS_b}{dz'}\right) = \kappa_S\nabla^2 S', \tag{19}$$

$$\rho' = \rho_R(\beta_2 S' - \beta_1 T'), \tag{20}$$

where $\mathbf{k}$ is the unit vector in the positive z-direction, $v = (\mu_f/\rho_R)$ (kinematic viscosity). The value of γ and δ is set equal to one in further analysis for simplicity.

The boundary conditions for the perturbed velocity, temperature and solute concentration are given by

$$w' = \frac{\partial^2 w'}{\partial z^2} = T' = S' = 0 \quad \text{at } z = 0, d. \tag{21}$$

The boundary conditions on velocity are stress-free conditions.

We eliminate p' from Eq. (17) and render the resulting equation and Eqs. (18) and (19) dimensionless by using the non-dimensional variables

$$(x', y', z') = (x^*, y^*, z^*)d, \quad w' = \left(\frac{vd}{k_z}\right)w^*,$$

$$t = \left(\frac{k_z}{v}\right)t^*, \quad T' = (\Delta T)T^*,$$

$$S' = (\Delta S)S^*, \quad \Omega = \left(\frac{\kappa_T}{d^2}\right)\omega, \tag{22}$$

to obtain linearized non-dimensional equations as (on dropping asterisks for simplicity)

$$\left[\left(\frac{\partial}{\partial t}+1\right)\nabla_1^2 + \left(\frac{\partial}{\partial t}+\frac{1}{\xi}\right)\frac{\partial^2}{\partial z^2} - MF\nabla^4\right]w = RFPr^{-1}\nabla_1^2 T - RsFPr^{-1}\nabla_1^2 S, \tag{23}$$

$$\left[\frac{\partial}{\partial t} - FPr^{-1}\nabla^2\right]T = -w\frac{\partial T_b}{\partial z}, \tag{24}$$

$$\left[\frac{\partial}{\partial t} - \tau FPr^{-1}\nabla^2\right]S = w, \tag{25}$$

where

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla^2 \equiv \nabla_1^2 + \frac{\partial^2}{\partial z^2}.$$

The dimensionless parameters that appear in the above equations are $R = \frac{\beta_1 g \Delta T d k_z}{v\kappa_T}$ (thermal Rayleigh number), $Rs = \frac{\beta_2 g \Delta S d k_z}{v\kappa_T}$ (solute Rayleigh number), $Pr = \frac{v}{\kappa_T}$ (Prandtl number), $M = \frac{\mu_e}{\mu_f}$ (viscosity ratio), $F = \frac{k_z}{d^2}$ (porous parameter), $\tau = \frac{\kappa_S}{\kappa_T}$ (diffusivity ratio), and $\xi = \frac{k_x}{k_z}$ (anisotropy parameter).

Combining Eqs. (23)–(25) we obtain an equation for the vertical component of velocity w in the form

$$\left\{\left[\left(\frac{\partial}{\partial t}+1\right)\nabla_1^2 + \left(\frac{\partial}{\partial t}+\frac{1}{\xi}\right)\frac{\partial^2}{\partial z^2} - MF\nabla^4\right] \times \left(\frac{\partial}{\partial t} - FPr^{-1}\nabla^2\right)\left(\frac{\partial}{\partial t} - \tau FPr^{-1}\nabla^2\right)\right\}w = -\left\{RFPr^{-1}\left(\frac{\partial}{\partial t} - \tau FPr^{-1}\nabla^2\right)\frac{\partial T_b}{\partial z}\nabla_1^2 + RsFPr^{-1}\left(\frac{\partial}{\partial t} - FPr^{-1}\nabla^2\right)\nabla_1^2\right\}w. \tag{26}$$

In dimensionless form, the velocity boundary conditions are

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = \cdots = 0 \quad \text{at } z = 0, 1. \tag{27}$$

In Eq. (26), $\frac{\partial T_b}{\partial z}$ is given by

$$\frac{\partial T_b}{\partial z} = -1 + \varepsilon f, \tag{28}$$

where $f = \mathrm{Re}\{(A(\lambda)e^{\lambda z} + A(-\lambda)e^{-\lambda z})e^{-i\omega t}\}$ with $\lambda = (1-i)\left(\frac{\omega}{2}\right)^{1/2}$ and $A(\lambda) = \frac{\lambda}{2}\left[\frac{e^{-i\phi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}\right]$.

3. Method of solution

We apply the perturbation technique to obtain the eigenfunctions w and eigenvalues R of Eq. (26) for the basic temperature distribution, which departs from the linear profile $(\partial T_b/\partial z = -1)$ by quantities of order ε. Thus, the eigenvalues of present problem differ from those of porous media analogue of two-component Benard convection problem by quantities of order ε. Since the adopted technique is based on small amplitudes, ε has to be less than unity. We therefore assume the solution of Eq. (26) in the form

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots, \tag{29}$$

$$R = R_0 + \varepsilon^2 R_2 + \cdots, \tag{30}$$

where R_0 is the critical Rayleigh number for the unmodulated double diffusive convection in an anisotropic porous medium.

Substituting Eqs. (29) and (30) into (26) and equating coefficients of like powers of ε, we obtain the following system of equations

$$Lw_0 = 0, \tag{31}$$

$$Lw_1 = FPr^{-1}(La_3)\left(-R_0 f\nabla_1^2 w_0\right), \tag{32}$$

$$Lw_2 = FPr^{-1}(La_3)\left(R_2\nabla_1^2 w_0 - R_0 f\nabla_1^2 w_1\right), \tag{33}$$

where

$$L \equiv (La_1)(La_2)(La_3) - R_0 FPr^{-1}(La_3)\nabla_1^2 + RsFPr^{-1}(La_2)\nabla_1^2$$

with

$$La_1 \equiv \left[\left(\frac{\partial}{\partial t}+1\right)\nabla_1^2 + \left(\frac{\partial}{\partial t}+\frac{1}{\xi}\right)\frac{\partial^2}{\partial z^2} - MF\nabla^4\right],$$

$$La_2 \equiv \left(\frac{\partial}{\partial t} - FPr^{-1}\nabla^2\right), \quad La_3 \equiv \left(\frac{\partial}{\partial t} - \tau FPr^{-1}\nabla^2\right)$$

and each w_0, w_1, w_2 is required to satisfy the boundary conditions of Eq. (27).

In Eq. (30) the odd powers of ε are missing because changing the sign of ε shifts the time origin only which does not affect the problem of stability and thus R should be independent of the sign of ε, i.e. $R_1, R_3, \ldots$, must be zero.

The eigenfunction w_0 is a solution of the problem with $\varepsilon = 0$, and solutions for this problem are $w_0^{(n)} = \exp\{i(lx + my)\}\sin(n\pi z)$, $n = 1, 2, 3, \ldots$, where l, m are the wave numbers in the xy-plane. The corresponding eigenvalues $R_0 = R_0^{(n)}$ are given by

$$R_0^{(n)} = \left[\frac{(\alpha^2 + n^2\pi^2)}{\alpha^2}\right]\left[\alpha^2 + \frac{\pi^2}{\xi} + MF(\alpha^2 + n^2\pi^2)^2\right] + \frac{Rs}{\tau}. \tag{34}$$

For a fixed value of α the least eigenvalue occurs for $n = 1$. R_0 assumes the minimum value for $\alpha = \alpha_c$ where α_c satisfies the equation

$$2MF(\alpha_c^2)^3 + (1 + 3MF\pi^2)(\alpha_c^2)^2 - (MF\pi^6 + (\pi^4/\xi)) = 0. \tag{35}$$

From Eq. (35), we note that the critical wave number α_c depends on anisotropy parameter ξ, the porous parameter F and viscosity ratio M.

The equation for w_1 then takes the form

$$Lw_1 = R_0 FPr^{-1}\alpha^2(La_3)f \sin \pi z. \tag{36}$$

Now let $La_3 \equiv (-\mathrm{i}\omega + \tau FPr^{-1}\alpha^2 - \tau FPr^{-1}D^2)$, where $D = \mathrm{d}/\mathrm{d}z$.

Thus

$$\begin{aligned}(La_3)f \sin \pi z = {} & \left[\tau FPr^{-1}(\alpha^2 + \pi^2) + \mathrm{i}\omega(\tau FPr^{-1} - 1)\right] \\ & \times f \sin \pi z - 2\tau F\lambda\pi Pr^{-1} f' \cos \pi z\end{aligned} \tag{37}$$

with

$$f' = \mathrm{Re}[(A(\lambda)\mathrm{e}^{\lambda z} - A(-\lambda)\mathrm{e}^{-\lambda z})\mathrm{e}^{-\mathrm{i}\omega t}].$$

Using Eq. (37), Eq. (36) becomes

$$Lw_1 = R_0 FPr^{-1}\alpha^2 \mathrm{Re}\left\{L_1 f \sin \pi z - 2\pi\lambda\tau FPr^{-1} f' \cos \pi z\right\}, \tag{38}$$

where $L_1 = \tau FPr^{-1}(\alpha^2 + \pi^2) + \mathrm{i}\omega(\tau FPr^{-1} - 1)$.

We solve Eq. (38) for w_1 by expanding the right hand side of it in Fourier series expansion and inverting the operator L. For this, we need the following Fourier series expansions

$$\begin{aligned}g_{nm}(\lambda) &= 2\int_0^1 \mathrm{e}^{\lambda z} \sin n\pi z \sin m\pi z \, \mathrm{d}z \\ &= -\frac{4nm\pi^2\lambda[1 + (-1)^{n+m+1}\mathrm{e}^{\lambda}]}{[\lambda^2 + (n+m)^2\pi^2][\lambda^2 + (n-m)^2\pi^2]},\end{aligned} \tag{39}$$

$$\begin{aligned}f_{nm}(\lambda) &= 2\int_0^1 \mathrm{e}^{\lambda z} \cos n\pi z \cos m\pi z \, \mathrm{d}z \\ &= -\frac{2\lambda[\lambda^2 + (n^2 + m^2)\pi^2][1 + (-1)^{n+m+1}\mathrm{e}^{\lambda}]}{[\lambda^2 + (n+m)^2\pi^2][\lambda^2 + (n-m)^2\pi^2]},\end{aligned} \tag{40}$$

so that

$$\mathrm{e}^{\lambda z} \sin m\pi z = \sum_{n=1}^{\infty} g_{nm} \sin n\pi z, \tag{41}$$

$$\mathrm{e}^{\lambda z} \cos m\pi z = \sum_{n=1}^{\infty} f_{nm} \cos n\pi z. \tag{42}$$

Let us now define

$$L(\omega, n) = \left(\omega^2 B_1 - B_3\right) - \mathrm{i}\omega B_2, \tag{43}$$

where

$$B_1 = \alpha^2 + \frac{n^2\pi^2}{\xi} + F(\alpha^2 + n^2\pi^2)^2(M + (1+\tau)Pr^{-1}),$$

$$\begin{aligned}B_2 = {} & \omega^2(\alpha^2 + n^2\pi^2) - FPr^{-1}(\alpha^2 + n^2\pi^2)\left(\alpha^2 + \frac{n^2\pi^2}{\xi}\right. \\ & \left. + MF(\alpha^2 + n^2\pi^2)^2\right)(1+\tau) - \tau(FPr^{-1})^2(\alpha^2 + n^2\pi^2)^3 \\ & + FPr^{-1}B_4 - Rs\alpha^2,\end{aligned}$$

$$\begin{aligned}B_3 = {} & (FPr^{-1})^2(\alpha^2 + n^2\pi^2)\left\{\tau(\alpha^2 + n^2\pi^2)\left(\alpha^2 + \frac{n^2\pi^2}{\xi}\right.\right. \\ & \left.\left. + MF(\alpha^2 + n^2\pi^2)^2\right)\right\} - \tau B_4 + Rs\alpha^2,\end{aligned}$$

$$B_4 = (\alpha^2 + \pi^2)\left[\alpha^2 + \frac{\pi^2}{\xi} + MF(\alpha^2 + \pi^2)^2\right] + \frac{Rs}{\tau}\alpha^2.$$

It is easily seen that

$$L(\sin n\pi z \mathrm{e}^{-\mathrm{i}\omega t}) = L(\omega, n) \sin n\pi z \mathrm{e}^{-\mathrm{i}\omega t},$$
$$L(\cos n\pi z \mathrm{e}^{-\mathrm{i}\omega t}) = L(\omega, n) \cos nn\pi z \mathrm{e}^{-\mathrm{i}\omega t}$$

and Eq. (38) now reads

$$\begin{aligned}Lw_1 = {} & R_0 FPr^{-1}\alpha^2 \mathrm{Re}\Big\{\sum L_1[A(\lambda)g_{n1}(\lambda) \\ & + A(-\lambda)g_{n1}(-\lambda)] \sin n\pi z \mathrm{e}^{-\mathrm{i}\omega t} - 2\pi\lambda\tau FPr^{-1} \\ & \times \sum[A(\lambda)f_{n1}(\lambda) + A(-\lambda)f_{n1}(-\lambda)] \cos n\pi z \mathrm{e}^{-\mathrm{i}\omega t}\Big\},\end{aligned} \tag{44}$$

so that

$$\begin{aligned}w_1 = {} & R_0 FPr^{-1}\alpha^2 \mathrm{Re}\left\{L_1 \sum \frac{A_n(\lambda)}{L(\omega, n)} \sin n\pi z \mathrm{e}^{-\mathrm{i}\omega t}\right. \\ & \left. - 2\pi\lambda\tau FPr^{-1} \sum \frac{B_n(\lambda)}{L(\omega, n)} \cos n\pi z \mathrm{e}^{-\mathrm{i}\omega t}\right\},\end{aligned} \tag{45}$$

where $A_n(\lambda) = A(\lambda)g_{n1}(\lambda) + A(-\lambda)g_{n1}(-\lambda)$, $B_n(\lambda) = A(\lambda) f_{n1}(\lambda) - A(-\lambda)f_{n1}(-\lambda)$.

To simplify Eq. (33) for w_2, we need

$$(La_3)fw_1 = L_n fw_1 - 2\tau FPr^{-1} DfDw_1, \tag{46}$$

where $L_n = \tau FPr^{-1}(\alpha^2 + n^2\pi^2) + \mathrm{i}\omega(\tau FPr^{-1} - 1)$.

The equation for w_2 then can be written as

$$\begin{aligned}Lw_2 = {} & -R_2\tau(FPr^{-1})^2\alpha^2(\alpha^2 + \pi^2) \sin \pi z \\ & + R_0 FPr^{-1}\alpha^2 \mathrm{Re}\left\{L_n fw_1 - 2\tau FPr^{-1} DfDw_1\right\}.\end{aligned} \tag{47}$$

We shall not require the solution of this equation but merely use it to determine R_2.

The solubility condition requires that the time-independent part of the right hand side of Eq. (47) must be orthogonal to $\sin \pi z$. Multiplying Eq. (47) by $\sin \pi z$ and integrating between 0 and 1 we obtain

$$\begin{aligned}R_2 = {} & \left[\frac{2R_0}{\tau FPr^{-1}(\alpha^2 + \pi^2)}\right] \mathrm{Re}\left\{L_n \int_0^1 \overline{fw_1} \sin \pi z \, \mathrm{d}z\right. \\ & \left. - 2\tau FPr^{-1} \int_0^1 \overline{DfDw_1} \sin \pi z \, \mathrm{d}z\right\},\end{aligned} \tag{48}$$

where an over bar denotes the time average.

We have the Fourier series expansions

$$f\sin\pi z = \mathrm{Re}\sum A_n(\lambda)\sin n\pi z\mathrm{e}^{-\mathrm{i}\omega t},$$
$$Df\sin\pi z = \mathrm{Re}\sum \lambda C_n(\lambda)\sin n\pi z\mathrm{e}^{-\mathrm{i}\omega t}, \tag{49}$$

where $C_n(\lambda) = A(\lambda)g_{n1}(\lambda) - A(-\lambda)g_{n1}(-\lambda)$.

We also note that the time average of product of two complex functions A and B is given by

$$\overline{A\cdot B} = \frac{1}{2\pi}\int_0^{2\pi} AB\,\mathrm{d}t = \frac{1}{2}A^*B = \frac{1}{2}AB^*, \tag{50}$$

where $*$ denotes a complex conjugate.

Using Eqs. (49) and (50) in Eq. (48) we obtain

$$R_2 = (La_4)\mathrm{Re}\left\{\sum_{n=1}^{\infty}\frac{L_1L_n^*|A_n(\lambda)|^2L^*(\omega,n)}{|L(\omega,n)|^2} - 4\pi^2|\lambda^2|^2(\tau FPr^{-1})^2\sum_{n=1}^{\infty}\frac{nB_n(\lambda)L^*(\omega,n)C_n^*(\lambda)}{|L(\omega,n)|^2}\right\}, \tag{51}$$

where

$$La_4 = \frac{R_0^2\alpha^2}{2\tau FPr^{-1}(\alpha^2+\pi^2)}.$$

Eq. (47) can now be solved for w_2, and the procedure may be continued to obtain further corrections to w and R.

We need the real part of $(L_1L_n^*L^*)$ which can be easily calculated

$$\mathrm{Re}\{L_1L_n^*L^*\} = (\omega^2B_1 - B_3)B_5 + \omega B_2B_6, \tag{52}$$

$$|L(\omega,n)|^2 = (\omega^2B_1 - B_3)^2 + \omega^2B_2^2, \tag{53}$$

$$|A_n(\lambda)|^2 = \frac{16\pi^4n^2\omega^2}{\{\omega^2+(n+1)^4\pi^4\}\{\omega^2+(n-1)^4\pi^4\}}, \tag{54}$$

where

$$B_5 = (\tau FPr^{-1})^2(\alpha^2+\pi^2)(\alpha^2+n^2\pi^2) + \omega^2(\tau FPr^{-1}-1)^2,$$
$$B_6 = \omega\tau FPr^{-1}(\tau FPr^{-1}-1)((\alpha^2+\pi^2)-(\alpha^2+n^2\pi^2)).$$

Similarly we can also find real part of $(B_n(\lambda)L^*(\omega,n)C_n(\lambda))$ easily.

4. Minimum Rayleigh number for convection

The value of the thermal Rayleigh number R obtained by this procedure is the eigenvalue corresponding to the eigenfunction w which, though oscillating remains bounded in time. Since R is a function of

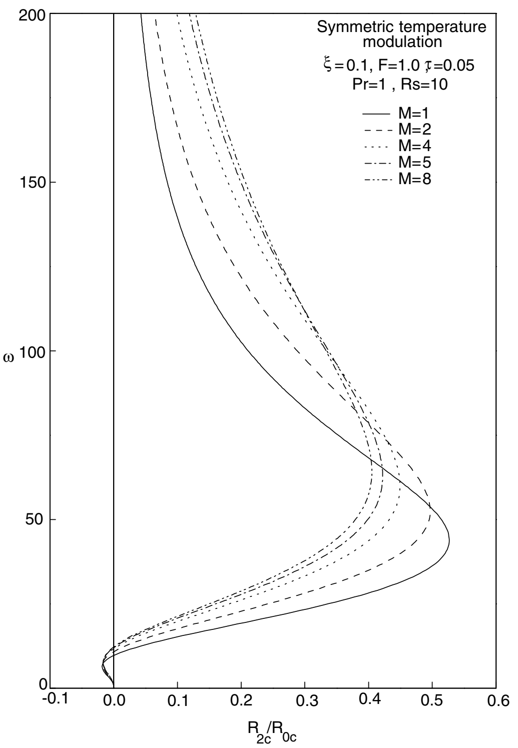

Fig. 1. Variation of R_{2c} with ω for different values of the viscosity ratio M.

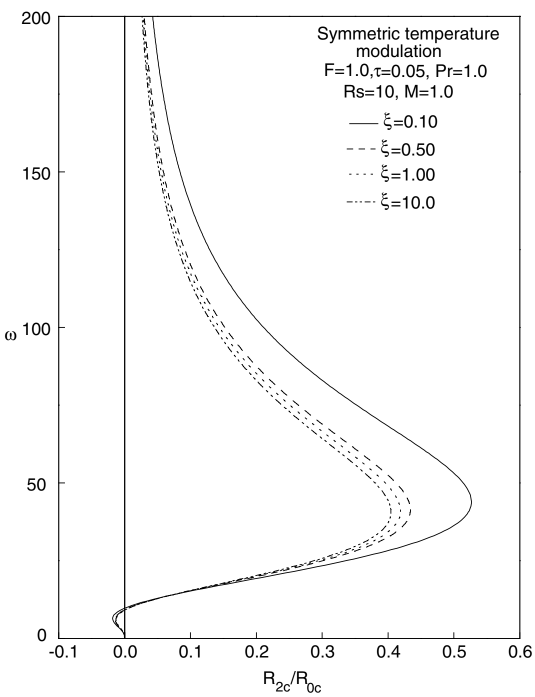

Fig. 2. Variation of R_{2c} with ω for different values of the anisotropy parameter ξ.

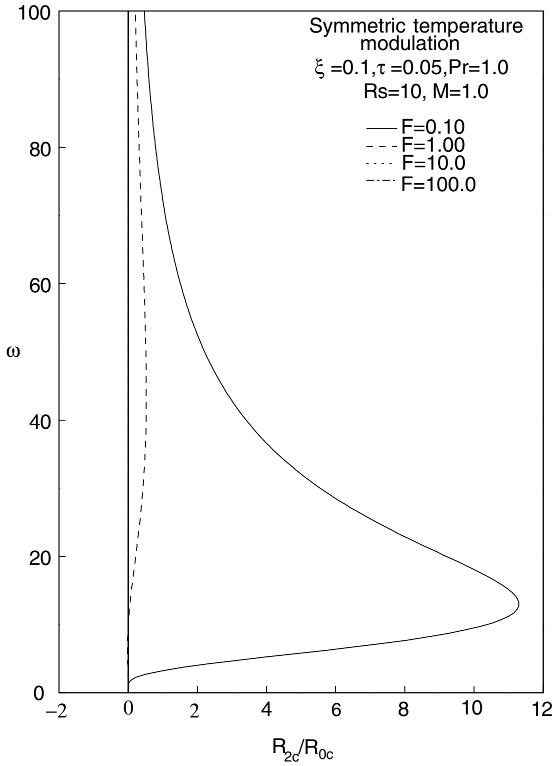

Fig. 3. Variation of R_{2c} with ω for different values of the porous parameter F.

the horizontal wave number α and amplitude of perturbation ε, we expand

$$R(\alpha,\varepsilon) = R_0(\alpha) + \varepsilon^2 R_2(\alpha) + \cdots, \tag{55}$$

$$\alpha = \alpha_0 + \varepsilon^2\alpha_2 + \cdots. \tag{56}$$

The critical value of the Rayleigh number R is computed up to $O(\varepsilon^2)$ by evaluating R_0 and R_2 at $\alpha = \alpha_0$. It is only when one wishes to evaluate R_4 that α_2 must be taken into account where $\alpha = \alpha_2$ minimizes R_4. To evaluate the critical value of R_2 denoted by R_{2c} we substitute $\alpha = \alpha_0$ in R_2, where α_0 is the value at which R_0 given by Eq. (34) is minimum. We evaluate R_{2c} for the following cases,

(a) when the oscillating temperature field is symmetric so that the wall temperatures are modulated in phase (with $\phi = 0$),
(b) when the wall temperature field is antisymmetric corresponding to out-of-phase modulation (with $\phi = \pi$),
(c) when only the temperature of the bottom wall is modulated, the upper wall being held at a constant temperature (with $\phi = -\mathrm{i}\infty$).

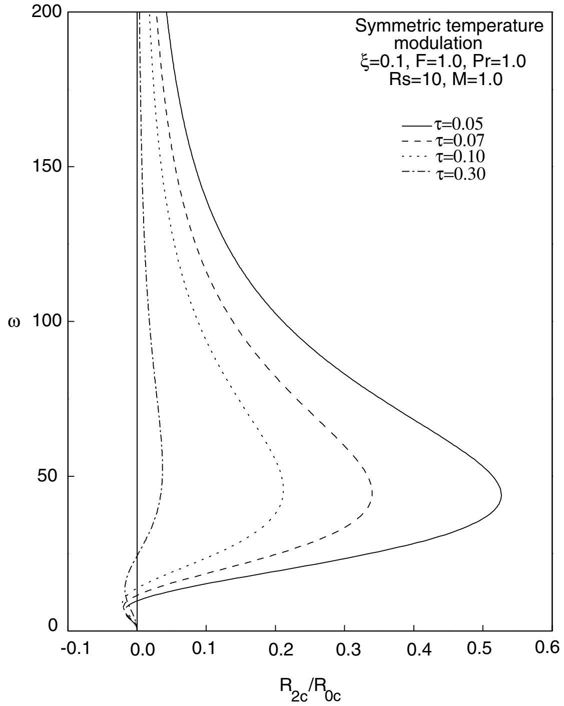

Fig. 4. Variation of R_{2c} with ω for different values of the diffusivity ratio τ.

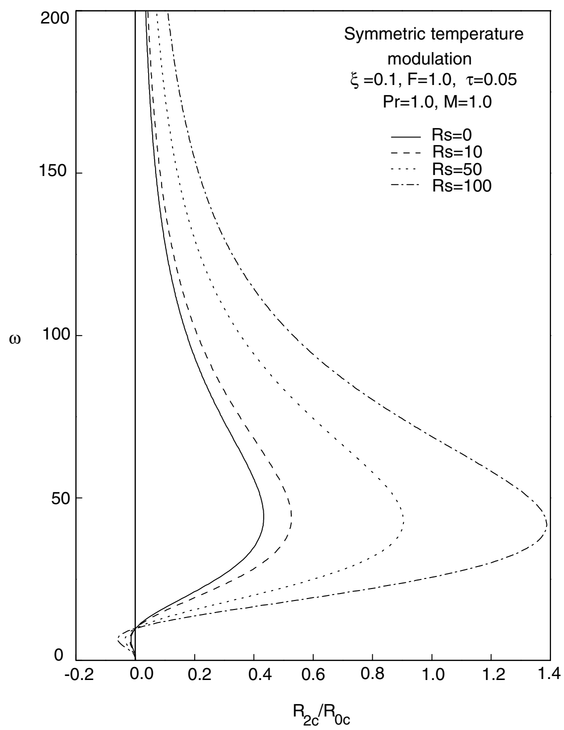

Fig. 5. Variation of R_{2c} with ω for different values of the solute Rayleigh number Rs.

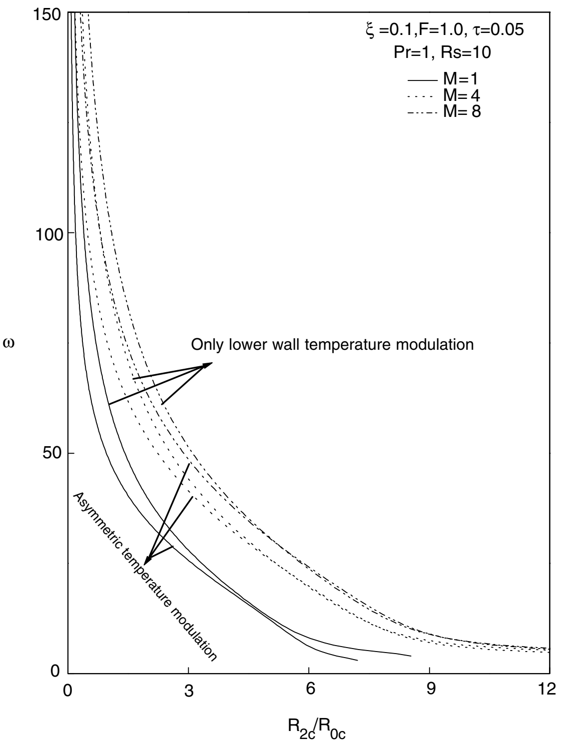

Fig. 6. Variation of R_{2c} with ω for different values of the viscosity ratio M.

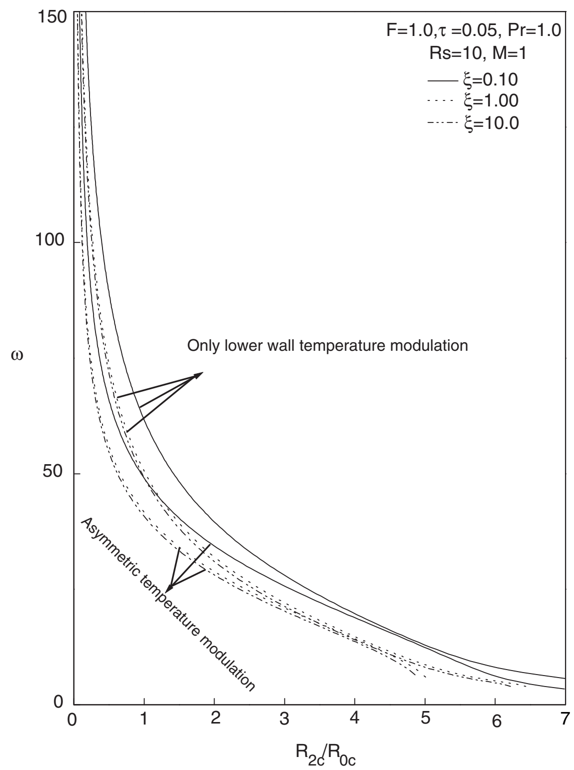

Fig. 7. Variation of R_{2c} with ω for different values of the anisotropy parameter ξ.

In Eq. (51) the sum extends over even values of n for case (a), odd values of n for case (b) and all integer values for case (c). The infinite series (51) converges in all cases.

The variation of R_{2c} with ω for different values of the parameters are depicted in Figs. 1–12 and the results are discussed in Section 5.

5. Results and discussion

The effect of time-periodic temperature modulation on the onset of double diffusive convection in a horizontal anisotropic porous layer is investigated using the linear stability analysis proposed by Venezian [5].

Fig. 1 shows the variation of R_{2c} with ω, for different values of the viscosity ratio M for the case of symmetric modulation of the wall temperature. We observe from this figure that, for small frequencies R_{2c} is negative indicating that the symmetric modulation has destabilizing effect while for moderate and large values of frequency its effect is stabilizing. The peak value of R_{2c} occurs around $\omega = 45$ and it depends on the viscosity ratio M. The effect of increasing viscosity ratio is to reduce the influence of modulation for small and moderate frequencies while for large frequencies its effect is stabilizing.

The effect of the anisotropy parameter ξ on the stability of the system for the case of symmetric modulation is shown in Fig. 2. R_{2c} is found to be positive over a wide range of values of the frequency ω. We find from this figure that as the anisotropy parameter ξ increases, the value of R_{2c} decreases indicating that the effect of increasing ξ is to reduce the effect of modulation. It is also found that the effect of ξ is insignificant for low frequencies. The peak value of R_{2c} occurs around $\omega = 42$ and its value depends on ξ.

The effect of porous parameter F on the stability of the system in presence of symmetric modulation is shown in Fig. 3. We observe that, as F increases the value of R_{2c} becomes small indicating that the large value of F reduces the effect of modulation. The curves for $F = 10$ and 100 almost coincide with $R_{2c} = 0$ line. This is due to the fact that Darcy resistance effect is dominant over the modulation effect.

Fig. 4 depicts the variation of R_{2c} with frequency ω, for different values of the diffusivity ratio τ for the case of symmetric modulation. It can be seen that an increase in the value of diffusivity ratio decreases the value of R_{2c} indicating that, the effect of increasing τ is to reduce the

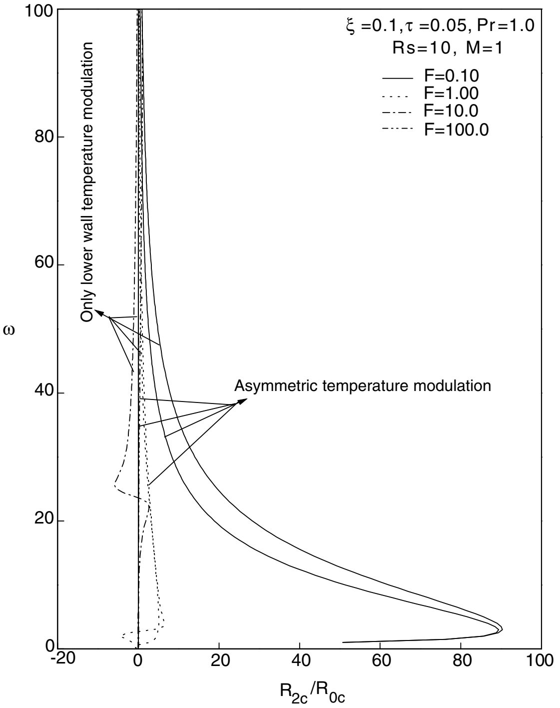

Fig. 8. Variation of R_{2c} with ω for different values of the porous parameter F.

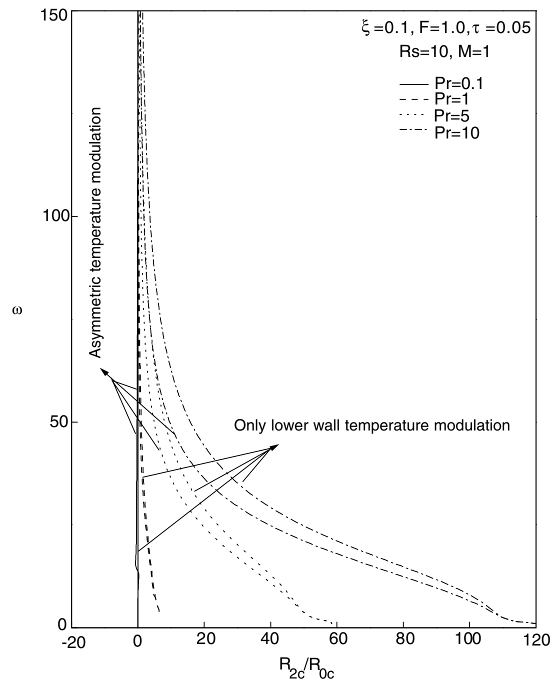

Fig. 10. Variation of R_{2c} with ω for different values of the Prandtl number Pr.

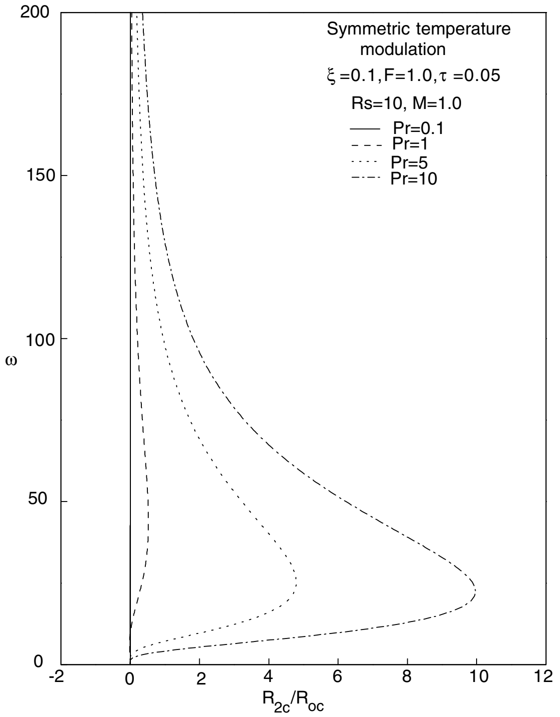

Fig. 9. Variation of R_{2c} with ω for different values of the Prandtl number Pr.

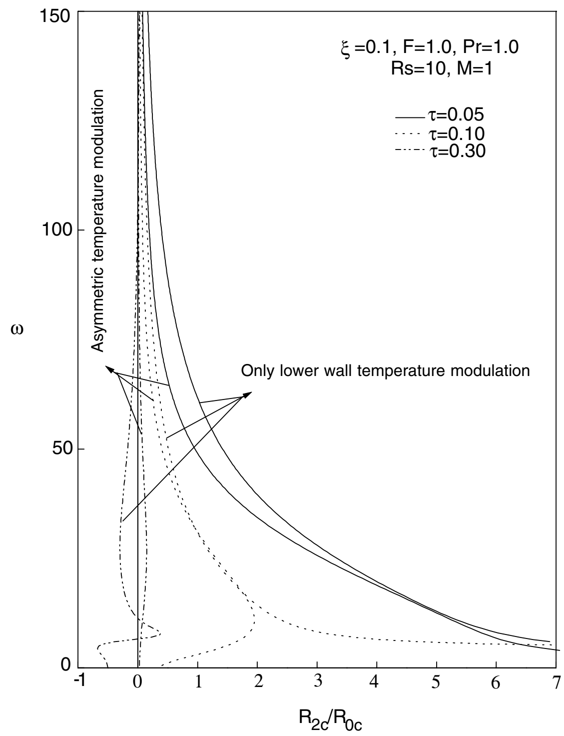

Fig. 11. Variation of R_{2c} with ω for different values of the diffusivity ratio τ.

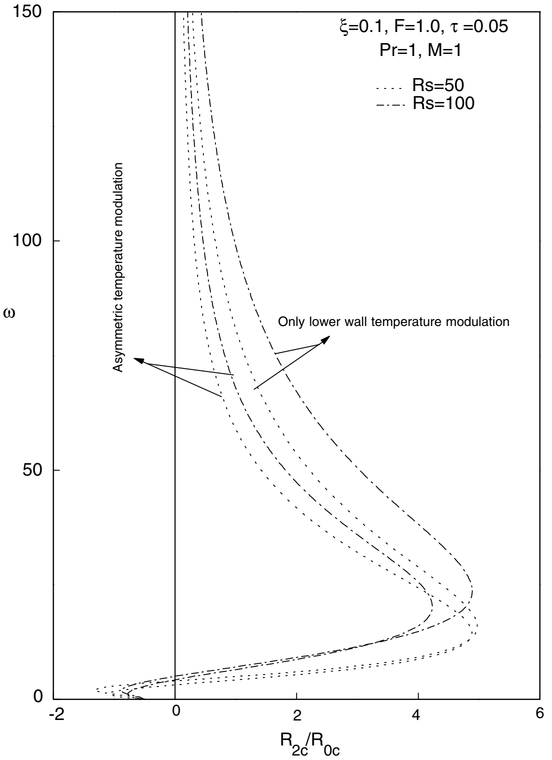

Fig. 12. Variation of R_{2c} with ω for different values of the solute Rayleigh number *Rs*.

effect of thermal modulation. The peak value of R_{2c} occurs at $\omega = 47$, and is independent of the diffusivity ratio τ.

Fig. 5 is the plot of R_{2c} versus ω for different values of the solute Rayleigh number *Rs* with respect to symmetric modulation of the wall temperature. We notice that the value of R_{2c} increases with an increase in the value of *Rs*, indicating that the effect of increasing solute Rayleigh number *Rs* is to delay the onset of convection. The peak value of R_{2c} occurs at $\omega = 42$ and is also independent of the solute Rayleigh number *Rs*.

The effect of viscosity ratio M on R_{2c} for the case of asymmetric modulation and only lower wall temperature is modulated is shown in Fig. 6. We observe that, the effect is stabilizing over the whole range of the frequencies. An increase in the value of M increases the value of R_{2c}, indicating that the effect of increasing the viscosity ratio is to make the system more stable.

The effect of anisotropy parameter ξ on the onset of convection in presence of asymmetric and only lower wall temperature modulation is shown in Fig. 7. We find that the effect of an increase in the value of ξ is to reduce the effect of modulation. The effect of ξ is qualitatively same in three types of modulation considered in this paper.

The effect of porous parameter F on the stability of the system in presence of asymmetric and only lower wall temperature modulation is shown in Fig. 8. We find that, an increase in the values of F decreases R_{2c}. Further we also find that as F increases beyond the value of one, R_{2c} become negative for some frequencies.

The effect of Prandtl number *Pr* on the onset of convection in presence of symmetric, asymmetric and only lower wall temperature modulation is shown in Figs. 9 and 10. We find that an increase in the value of *Pr* increases R_{2c}. Thus the large Prandtl number fluid systems are more stable in the presence of thermal modulation.

Fig. 11 depicts the variation of R_{2c} with ω, for different values of diffusivity ratio τ for both asymmetric and lower wall temperature modulation. We find that, the effect of increasing the value of τ is to reduce the effect of asymmetric modulation, while in case of only wall temperature modulation its effect is destabilizing.

The variation of the shift in the Rayleigh number R_{2c} with frequency ω for different values of the solute Rayleigh number *Rs* is shown in Fig. 12. From this figure we observe that, for small frequencies, R_{2c} decreases with an increase in the value of the solute Rayleigh number, indicating that its effect is destabilizing. On the other hand for $\omega \geqslant 20$ the effect of increasing solute Rayleigh number is found to be stabilizing. However for very small values of the frequency, R_{2c} is negative, indicating that the effect is destabilizing one.

The results of the asymmetric and lower wall temperature modulation are found to be qualitatively similar. It is observed that for large frequencies the effect of modulation disappears.

6. Conclusions

Three types of thermal modulation effect on the onset of double diffusive convection in an anisotropic porous layer has been studied in this paper and the following conclusions are drawn:

1. Low frequency symmetric modulation is destabilizing while high frequency symmetric modulation is always stabilizing.
2. Asymmetric modulation and only lower wall temperature modulation is stabilizing for all frequencies. However some additional parameters like porous parameter, Prandtl number, diffusivity ratio, may influence the stability of the system.
3. The effect of Prandtl number is found to be stabilizing and the large Prandtl number fluid systems are more stable in the presence of thermal modulation.
4. The effect of the anisotropy parameter ξ in case of symmetric modulation is significant for moderate values of the frequency. However its effect is insignificant for low frequencies. The effect of increasing ξ

is to reduce the effect of thermal modulation in all types of modulations considered.
5. The effect of increasing porous parameter is to reduce the effect of modulation. However in case of only lower wall temperature modulation, large F has destabilizing effect.
6. The effect of increasing solute Rayleigh number is to stabilize the system in general. However in case of only lower wall temperature modulation, it destabilizes the system for low frequencies.
7. The effect of increasing diffusivity ration τ is to reduce the effect of thermal modulation.

The results of this study indicate that imposed time-periodic boundary temperatures can give rise to sub-critical or super critical motions. The problem throws light on an external means of controlling double diffusive convection (either advancing or delaying) in an anisotropic porous medium.

Acknowledgements

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References

[1] N. Rudraiah, P.K. Srimani, R. Friedrich, Finite amplitude convection in a two component fluid saturated porous layer, Heat Mass Transfer 25 (1982) 715–722.
[2] D. Poulikakos, Double diffusive convection in a horizontally sparsely packed porous layer, Int. Commun. Heat Mass Transfer 13 (1986) 587–598.
[3] N. Rudraiah, M.S. Malashetty, The influence of coupled molecular diffusion on double diffusive convection in a porous medium, ASME J. Heat Transfer 108 (1986) 872–876.
[4] N. Rudraiah, M.S. Malashetty, Effect of modulation on the onset of convection in a sparsely packed porous layer, ASME J. Heat Transfer 122 (1990) 685–689.
[5] S.H. Davis, The stability of time periodic flows, Annu. Rev. Fluid Mech. 8 (1976) 57–74.
[6] G. Venezian, Effect of modulation on the onset of thermal convection, J. Fluid Mech. 35 (1969) 243–254.
[7] S. Rosenblat, D.M. Herbert, Low frequency modulation of thermal instability, J. Fluid Mech. 43 (1970) 385–389.
[8] S. Rosenblat, G.A. Tanaka, Modulation of thermal convection instability, Phys. Fluids 14 (1971) 1319–1322.
[9] M.H. Roppo, S.H. Davis, S. Rosenblat, Benard convection with time periodic heating, Phys. Fluids 27 (1984) 796–803.
[10] J.P. Caltagirone, Stabilite d'une couche poreuse horizontale soumise a des conditions aux limite periodiques, Int. J. Heat Mass Transfer 19 (1976) 815–820.
[11] M.S. Malashetty, V.S. Wadi, Rayleigh–Benard convection subject to time dependent wall temperature in a fluid saturated porous layer, Fluid Dyn. Res. 24 (1999) 293–308.
[12] M.S. Malashetty, D. Basavaraja, Rayleigh–Benard convection subject to time dependent wall temperature/gravity in a fluid-saturated anisotropic porous medium, Heat Mass Transfer 38 (2002) 551–563.
[13] M.S. Malashetty, D. Basavaraja, The effect of thermal/gravity modulation on the onset of convection in a horizontal anisotropic porous layer, Int. J. Appl. Mech. Eng. 8 (3) (2003) 425–439.
[14] D.A. Nield, A. Bejan, Convection in Porous Media, second ed., Springer-Verlag, New York, 1999.